Curl of a Vector

Consider a vector **A** (we use **boldface** to denote a vector) with rectangular coordinates A_x, A_y, and A_z. We can write **A** as follows:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

where **i** is a unit vector in the x direction, **j** is a unit vector in the y direction, and **k** is a unit vector in the z direction.

Using $\nabla = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$, we define the *curl* of **A**, written $\nabla \times \mathbf{A}$ ("del cross A") or **curl A**, as follows:

$$\nabla \times \mathbf{A} = (\partial A_z/\partial y - \partial A_y/\partial z)\,\mathbf{i} - (\partial A_z/\partial x - \partial A_x/\partial z)\,\mathbf{j} + (\partial A_y/\partial x - \partial A_x/\partial y)\,\mathbf{k} \qquad (1)$$

(For a quick review of partial derivatives, see http://www.math.wisc.edu/~CONRAD/s08/partials.pdf)

This is an ugly formula, so it is often written as a determinant:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

(For a quick review of how to calculate determinants, see http://www.ucl.ac.uk/mathematics/geomath/level2/mat/mat121.html)

Notice that the curl is a *vector*. (It can be thought of as a cross product between the del operator – the ∇ – and the vector **A**, but understanding that point is not essential to success here.)

The following websites may also be helpful:

http://www.tech.plym.ac.uk/maths/resources/pdflatex/div_curl.pdf
http://hyperphysics.phy-astr.gsu.edu/hbase/vvec.html

***Simple* EXAMPLE:**

Suppose $\mathbf{A} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$. This is a nice, simple, constant vector. Since all the components are constants, all the partial derivatives of the components are zero.

$$\nabla \times \mathbf{A} = 0$$

***More Complicated* EXAMPLE:**

Suppose $\mathbf{A} = 3x^2\,\mathbf{i} + 5xyz\,\mathbf{j}$. Calculate **curl A**.

Before we start the formula, notice that in this example, $A_z = 0$, so several of the terms drop out.

$$\textbf{curl A} = (\partial A_z/\partial y - \partial A_y/\partial z)\,\mathbf{i} - (\partial A_z/\partial x - \partial A_x/\partial z)\,\mathbf{j} + (\partial A_y/\partial x - \partial A_x/\partial y)\,\mathbf{k}$$

$$= (\partial/\partial y(0) - \partial/\partial z(5xyz))\,\mathbf{i} - (\partial/\partial x(0) - \partial/\partial z(3x^2))\,\mathbf{j} + (\partial/\partial x(5xyz) - \partial/\partial y(3x^2))\,\mathbf{k}$$

$$= -5xy\,\mathbf{i} + 5yz\,\mathbf{k}$$

If you want to know the value of the curl at any point (x, y, z), you just substitute the values of x, y, and z into the curl formula.

To find the curl of the function above at the point x = 4, y = 6, z = 9, substitute those values into the curl formula for that function:

$$\textbf{curl A} = \nabla \times \mathbf{A} = -5xy\,\mathbf{i} + 5yz\,\mathbf{k} = -5(4)(6)\,\mathbf{i} + 5(6)(9)\,\mathbf{k} = -120\,\mathbf{i} + 270\,\mathbf{k}$$

Interpretation

The curl represents how much a vector field "swirls" around a point, and indicates the direction of the "swirl."

For a nice concise discussion of the physical interpretation of curl, see
http://keep2.sjfc.edu/faculty/kgreen/vector/block2/del_op/node9.html or
http://betterexplained.com/articles/vector-calculus-understanding-circulation-and-curl/.

Also see http://mathinsight.org/curl_idea for an interactive visual.