Curvature and Acceleration in 3 Dimensions (Calc 3)

\[ \mathbf{r}(t) = <x(t), y(t), z(t)>, \quad \mathbf{r}'(t) = \mathbf{v}(t) = <x'(t), y'(t), z'(t)>, \quad \mathbf{v}(t) = |\mathbf{v}(t)|, \]
\[ \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \]

1. **Arc Length** \( s \):
   \[ s(t) = \int_a^b \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} \, du = \int_a^b |\mathbf{r}'(u)| \, du \]
   a. From \( t = a \) to \( t = b \):
      \[ s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt = \int_a^b \mathbf{v}(t) \, dt \]
   c. So \( \frac{ds}{dt} = |\mathbf{r}'(t)| = \mathbf{v}(t) \)

2. **Unit Tangent Vector** \( \mathbf{T} \):
   \[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \]

3. **Curvature** \( \kappa \):
   a. \[ \kappa = \frac{\left| \frac{dT}{ds} \right|}{\left| \mathbf{r}'(t) \right|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r'}(t)|^3} \]
   b. For a function \( y = f(x) \):
      \[ \kappa = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} \]

4. **Principal Unit Normal Vector** \( \mathbf{N} \):
   \[ \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \]
   a. \( \mathbf{N} \) is perpendicular to the unit tangent vector \( \mathbf{T} \) and points in the direction in which the curve is bending.
   b. Using acceleration:
      \[ \mathbf{N} = \frac{\mathbf{a} - a_s \mathbf{T}}{a_n} \]

5. **Binormal Vector**: \( \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \)

6. **Normal and Osculating Planes**:
   a. The **normal plane** is the plane that contains \( \mathbf{N} \) and \( \mathbf{B} \).
   b. The **osculating plane** contains \( \mathbf{T} \) and \( \mathbf{N} \).
   c. The circle in the osculating plane that most approximates the curve at the point \( \mathbf{C} \) (same curvature and tangent and its center lies along \( \mathbf{N} \)) is called the **osculating circle**. Its radius \( \rho \) is called the **radius of curvature** and is \( \rho = \frac{1}{\kappa} \), the reciprocal of the curvature of the curve at point \( \mathbf{C} \).
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7. Acceleration: \( a = \frac{dv}{dt} T + \kappa v^2 N = a_T T + a_N N \)

a. Tangential Component of Acceleration: 
\[
  a_T = \frac{dv}{dt} = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}
\]

b. Normal Component of Acceleration: 
\[
  a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}
\]