Curvature and Acceleration in 3 Dimensions (Calc 3)

$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $\mathbf{r}'(t) = \mathbf{v}(t) = \langle x'(t), y'(t), z'(t) \rangle$, $v(t) = |\mathbf{v}(t)|$,
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

1. **Arc Length** s: $s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\, du = \int_a^t |\mathbf{r}'(u)|\, du$

   a. From $t = a$ to $t = b$: $s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b v(t)\, dt$

   c. So $\dfrac{ds}{dt} = |\mathbf{r}'(t)| = v(t)$

2. **Unit Tangent Vector T**: $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

3. **Curvature** κ:

   a. $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

   b. For a function $y = f(x)$: $\kappa = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

4. **Principal Unit Normal Vector N**: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

   a. **N** is perpendicular to the unit tangent vector **T** and points in the direction in which the curve is bending.

   b. **Using acceleration:** $\mathbf{N} = \dfrac{\mathbf{a} - a_T \mathbf{T}}{a_N}$

5. **Binormal Vector**: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

6. **Normal and Osculating Planes**:

   a. The **normal plane** is the plane that contains **N** and **B**.

   b. The **osculating plane** contains **T** and **N**.

   c. The circle in the osculating plane that most approximates the curve at the point C (same curvature and tangent and its center lies along **N**) is called the **osculating circle**. Its radius ρ is called the **radius of curvature** and is $\rho = \dfrac{1}{\kappa}$ the reciprocal of the curvature of the curve at point C.

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7. **Acceleration a**: $\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N} = a_T\mathbf{T} + a_N\mathbf{N}$

   a. **Tangential Component of Acceleration**: $a_T = \frac{dv}{dt} = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$

   b. **Normal Component of Acceleration**: $a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$