Graphing Rational Functions

$$f(x) = \frac{N(x)}{D(x)} \quad \textit{where N and D are polynomials with no common factors.}$$

I. **V.A.:** Find the **vertical asymptote(s)** by setting the denominator equal to zero: ***D(x)* = 0**

II. **H.A.:** Find the **horizontal asymptote** by using the following rules:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots}{b_m x^m + a_{m-1}x^{m-1} + \cdots}$$ The degree of the numerator is n. The degree of denominator is m.

a) If n<m, then y=0 (x-axis) is the horizontal asymptote
b) If n=m, then $y = \frac{a_n}{b_m}$ is the horizontal asymptote.
c) If n>m, then there is no horizontal asymptote
Exception: If n>m by 1, then the graph has a **slant asymptote** (S.A.)

III. Find the **x-intercepts** by setting the numerator equal to zero and solve. ***N(x)* = 0**

IV. Find the **y-intercepts** by substituting zero for x in the function, $f(0)$

V. Make a table of values between and beyond each x-intercept and vertical asymptote; ***more values provide a more accurate graph.***

VI. Sketch the graph by using the information obtained.

. ***NOTE: Remember that the graph never crosses the vertical asymptotes!!!!***

EXAMPLE 1: *Graph the function -* $f(x) = \frac{3x^2}{x^2-4}$

I. **V.A.:** Set the denominator equal to zero and solve.

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Vertical asymptotes at x = 2 & x = -2

II. **H.A.:** The degree of the numerator and denominator are both equal to 2. So the horizontal asymptote is $= \frac{3}{1}$.

Horizontal asymptote at y=3

III. ***x*-intercept(s):** Set the numerator equal to zero and solve.

$$3x^2 = 0$$
$$x^2 = 0$$
$$x = 0$$

Intercept at x = 0

IV. ***y*-intercept(s):** Substitute zero for *x* in the function and solve.

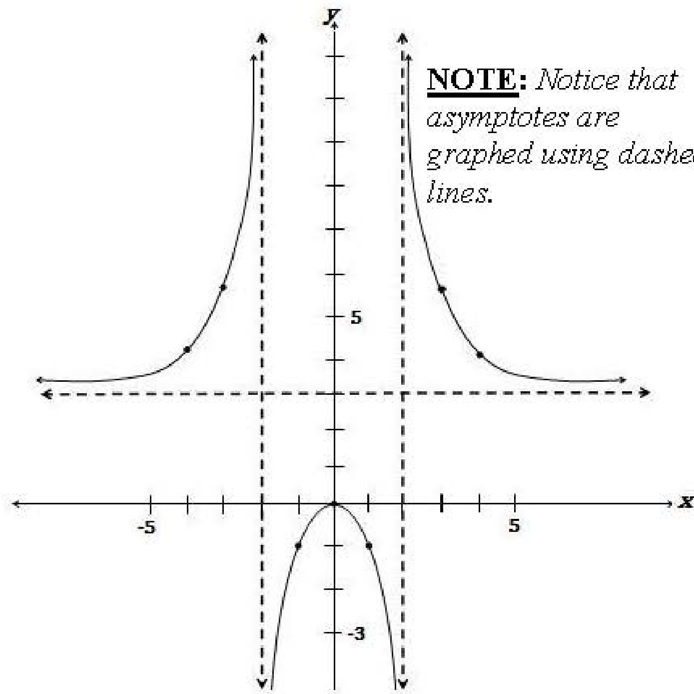
$$y = \frac{3(0)^2}{(0)^2 - 4} = -\frac{0}{4}$$
$$y = 0$$

Intercept at y = 0

V. **Table:** Make a table of values near each vertical asymptote and x-intercept and plot the values

x	$f(x) = \frac{3x^2}{x^2 - 4}$
-4	4
-3	$\frac{27}{5}$
-1	-1
1	-1
3	$\frac{27}{5}$
4	4

VI. **Graph:** Sketch the graph by using the information obtained.



NOTE: *Notice that asymptotes are graphed using dashed lines.*

EXAMPLE 2: *Graph the function -* $f(x) = \frac{X+3}{x^2-9}$

$$f(x) = \frac{X+3}{x^2-9} = \frac{\cancel{x+3}}{\cancel{(x+3)}(x-3)} = \frac{1}{(x-3)}$$

I. **V.A.:** $x - 3 = 0$ $\quad$ $\boldsymbol{x = 3}$

II. **H.A.:** $n < m$ *so* $\boldsymbol{y = 0}$

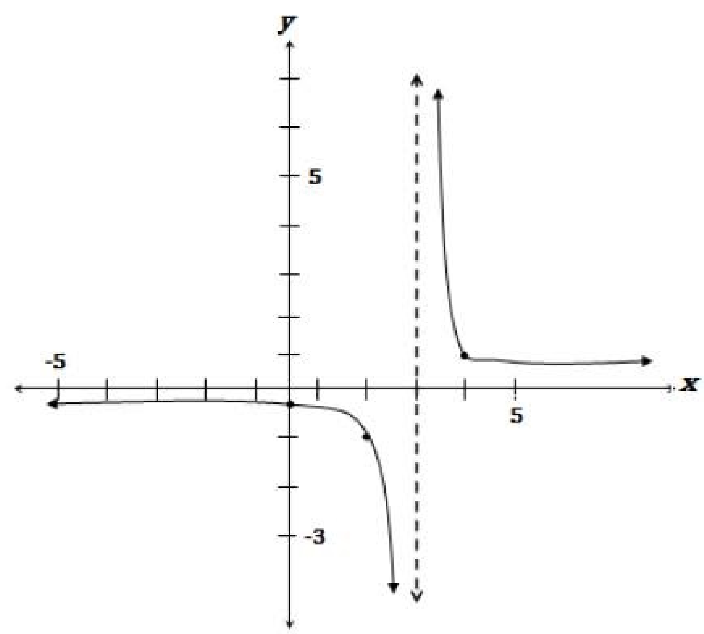
III. ***x*-intercept:** $1 \neq 0$, so there are none

IV. ***y*-intercept:** $\frac{1}{0-3} = -\frac{1}{3}$

V. **Table:**

x	$f(x) = \frac{x+3}{x^2-9}$
2	-1
4	1

VI. **Graph:**



EXAMPLE 3: *Graph the function -* $f(x) = \frac{x+3}{x^2+9}$

I. **V.A.:** $x^2 + 9 \neq 0$

The function is always positive, so there are no vertical asymptotes

II. **H.A.:** $n < m$ *so* $y = \mathbf{0}$

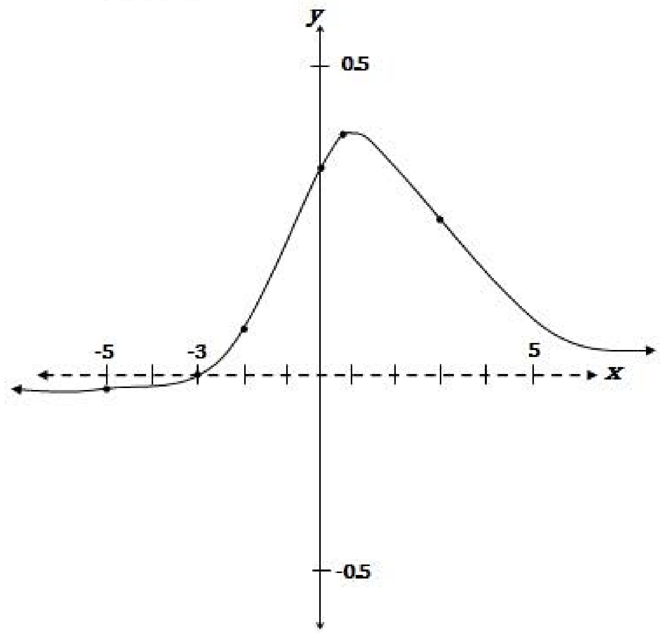
III. ***x*-intercept:** $x + 3 = 0$ $\quad \boldsymbol{x = -3}$

IV. ***y*-intercept:** $\frac{0+3}{0^2+9} = \frac{1}{3}$

V. **Table:**

x	-5	-2	1	3
$f(x) = \frac{x+3}{x^2+9}$	$-\frac{1}{17}$	$\frac{1}{13}$	$\frac{2}{5}$	$\frac{1}{3}$

VI. **Graph:**



EXAMPLE 4: *Graph the function -* $f(x) = \frac{x^2+1}{x-1}$

I. **V.A.:** $x - 1 = 0$ $\quad \boldsymbol{x = 1}$

II. **H.A.:** *n > m by 1, therefore there are no horizontal asymptotes, but there is a slant asymptote; the equation can be found by division*

$f(x) = \frac{x^2+1}{x-1} = \boxed{x + 1} + \frac{2}{x-1}$

Equation of Slant Asymptote
y = x+1

III. ***x*-intercept:** $x^2 + 1 = 0$ $\quad x^2 \neq -1$

so there are none

IV. ***y*-intercept:** $\frac{0^2+1}{0-1} = -1$

V. **Table:**

x	$-\frac{1}{2}$	2
$f(x) = \frac{x^2+1}{x-1}$	$-\frac{5}{6}$	5

VI. **Graph:**