

**Some Important Series**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1 \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} \quad \text{for } |x| \leq 1$$

**Binomial Series**

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{for } |x| < 1 \text{ where } \binom{k}{0} = 1 \text{ and } \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \text{ for } n \geq 1$$

**Geometric Series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for } |r| < 1, \text{ diverges for } |r| \geq 1$$

**Harmonic Series/p-Series**

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.} \quad \text{More generally } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p \leq 1 \text{ and converges if } p > 1$$

**Power Series Centered at a:**  $\sum_{n=0}^{\infty} c_n (x-a)^n$

**Taylor Series/Maclaurin Series:**

If a function  $f$  has a power series representation at  $a$ , then it can be written in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \text{This is called the Taylor series of } f \text{ at } a.$$

If  $a = 0$ , then this is a Maclaurin series:  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

**Convergence Tests**

**n-th Term Test:**  $\sum_{n=1}^{\infty} a_n$  diverges if  $\lim_{n \rightarrow \infty} a_n \neq 0$  or DNE

**Integral Test:**  $f(x)$  continuous, positive, decreasing for  $x \in [1, \infty)$  and  $a_n = f(n)$   
 $\Rightarrow$  If  $\int_1^{\infty} f(x)dx$  converges (diverges), then  $\sum_{n=1}^{\infty} a_n$  converges (diverges)

$a_n$  and  $b_n \geq 0$  for all  $n$ , then

**Comparison Test:** 1)  $\sum b_n$  converges and  $a_n \leq b_n$  for all  $n \Rightarrow \sum a_n$  converges  
 2)  $\sum b_n$  diverges and  $a_n \geq b_n$  for all  $n \Rightarrow \sum a_n$  diverges

**Limit Comparison Test:** If  $a_n, b_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ , ( $c$  finite)  
 then  $\sum a_n$  and  $\sum b_n$  both converge or both diverge

**Alternating Series Test:**  $b_n > 0$   
 $b_{n+1} \leq b_n$   
 $\lim_{n \rightarrow \infty} b_n = 0$  }  $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} b_n$  converges

**Absolute Convergence Test:**  $\sum |a_n|$  converges  $\Rightarrow \sum a_n$  converges

**Ratio Test:**

Let  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ . Then

**Root Test:**

Let  $a_n \geq 0$  and  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L$ . Then

$L < 1 \Rightarrow \sum a_n$  is absolutely convergent

$L < 1 \Rightarrow \sum a_n$  converges

$L > 1$  or  $L = \infty \Rightarrow \sum a_n$  diverges

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