Calculus II

Infinite Series

Some Important Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1 \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} \quad \text{for } |x| \le 1$$

Binomial Series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{for } |x| < 1 \text{ where } \binom{k}{0} = 1 \text{ and } \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \text{ for } n \ge 1$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for } |r| < 1, \quad \text{diverges for } |r| \ge 1$$

Harmonic Series/p-Series

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. More generally $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \le 1$ and converges if $p > 1$

***Power Series Centered at* a:** $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$

Taylor Series/Maclaurin Series:

If a function *f* has a power series representation at *a*, then it can be written in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is called the Taylor series of *f* at *a*.

If $a = 0$, then this is a Maclaurin series: $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

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Convergence Tests

n-th Term Test: $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$ or DNE

Integral Test: $f(x)$ continuous, positive, decreasing for $x \in [1, \infty)$ and $a_n = f(n)$

$\Rightarrow$ If $\int_1^{\infty} f(x)dx$ converges (diverges), then $\sum_{n=1}^{\infty} a_n$ converges (diverges)

Comparison Test: a_n and $b_n \geq 0$ for all n, then

1) $\sum b_n$ converges and $a_n \leq b_n$ for all $n \Rightarrow \sum a_n$ converges
2) $\sum b_n$ diverges and $a_n \geq b_n$ for all $n \Rightarrow \sum a_n$ diverges

Limit Comparison Test: If $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, ($c$ finite)

then $\sum a_n$ and $\sum b_n$ both converge or both diverge

Alternating Series Test:

$$\left.\begin{array}{l} b_n > 0 \\ b_{n+1} \leq b_n \\ \lim_{n\to\infty} b_n = 0 \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} b_n \text{ converges}$$

Absolute Convergence Test: $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges

Ratio Test:

Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$. Then

$L < 1 \Rightarrow \sum a_n$ is absolutely convergent
$L > 1$ or $L = \infty \Rightarrow \sum a_n$ diverges
$L = 1 \Rightarrow$ inconclusive

Root Test:

Let $a_n \geq 0$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = L$. Then

$L < 1 \Rightarrow \sum a_n$ converges
$L > 1$ or $L = \infty \Rightarrow \sum a_n$ diverges
$L = 1 \Rightarrow$ inconclusive