Vector Cross Product

Consider a vector **A** (we use **boldface** to denote a vector) with rectangular coordinates A_x, A_y, and A_z. We can write A as follows:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

where **i** is a unit vector in the x direction, **j** is a unit vector in the y direction, and **k** is a unit vector in the z direction.

If we have two vectors **A** and **B**, defined as above, then we define the *cross product* of **A** and **B**, written **A x B** (read "A cross B") as follows:

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\,\mathbf{i} - (A_xB_z - A_zB_x)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k} \qquad (1)$$

Notice that the cross product is a *vector*. It may also be referred to as the *vector product.*

This is an ugly formula, so it is often expressed as a determinant:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(*For a quick review of how to calculate determinants, see* *http://www.ucl.ac.uk/mathematics/geomath/level2/mat/mat121.html*)

The following websites may also be helpful:

http://mathworld.wolfram.com/CrossProduct.html
http://www.khanacademy.org/math/linear-algebra/v/linear-algebra--cross-product-introduction
http://hyperphysics.phy-astr.gsu.edu/hbase/vvec.html

EXAMPLE:

Suppose **A** = 3**i** + 5**j** and **B** = 4**i** – 2**j**. Compute **A x B**.

STEP 1: Identify the components of the individual vectors

$A_x = 3$ $A_y = 5$ $A_z = 0$ $B_x = 4$ $B_y = -2$ $B_z = 0$

STEP 2: Use equation (1) to compute the cross product

We could barrel through the formula, or we could notice that $A_z = B_z = 0$. That means the first two terms in the ugly formula will also be zero. So here,

$$\mathbf{A} \times \mathbf{B} = (A_xB_y - A_yB_x)\,\mathbf{k} = [(3)(-2) - (5)(4)]\,\mathbf{k} = (-6 - 20)\,\mathbf{k} = -26\,\mathbf{k}$$

Aha. In this example, the vectors **A** and **B** were all in the x-y plane, but the cross product **A x B** was all in the z direction. This leads us to an important point:

NOTE: The cross product **A x B** defines a vector that is *perpendicular* to **A** and **B**.

Alternate Formula

It happens that the magnitude of the cross product can also be calculated as follows:

$$|\,\mathbf{A} \times \mathbf{B}\,| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta \qquad (2)$$

where |**A**| is the magnitude of **A**, |**B**| is the magnitude of **B**, and θ is the angle between them. This formula may be helpful when you are given the magnitudes and angles to begin with.

EXAMPLE:

Suppose A is a vector of magnitude 6 at an angle of 60°. Suppose B is a vector of magnitude 4 at an angle of -25°. Compute | **A x B** |.

STEP 1: Compute the angle θ between the vectors

$$\theta = 60° - (-25°) = 85°$$

STEP 2: Compute the sine of the angle θ

$$\sin\theta = \sin(85°) \sim 0.996 \text{ (rounding off)}$$

STEP 3: Use equation (2) to compute the magnitude of the cross product

$$|\,\mathbf{A} \times \mathbf{B}\,| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta = (6)(4)(0.996) \sim 23.9$$

Note that | **A x B** | ranges from 0 when **A** and **B** are parallel to |**A**| |**B**| when **A** and **B** are perpendicular.