Vector Gradient of a Scalar Function

Consider a scalar function $f = f(x, y, z)$.

Using $\nabla = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$, we define the *gradient* of *f*, written ∇f or **grad** *f*, as follows:

$$\textbf{grad } f = \nabla f = \partial f/\partial x\,\mathbf{i} + \partial f/\partial y\,\mathbf{j} + \partial f/\partial z\,\mathbf{k} \qquad (1)$$

where **i** is a unit vector in the x direction, **j** is a unit vector in the y direction, and **k** is a unit vector in the z direction.

Notice that the gradient is a *vector*.

The following websites may also be helpful:

http://keep2.sjfc.edu/faculty/kgreen/vector/block2/del_op/node1.html#SECTION00010000000000000000
http://hyperphysics.phy-astr.gsu.edu/hbase/vvec.html

(For a quick review of partial derivatives, see http://www.math.wisc.edu/~CONRAD/s08/partials.pdf)

EXAMPLE:

Suppose $f = x^2y^3z^4$. Compute **grad** *f*.

STEP 1: Compute the necessary partial derivatives

$$\partial f/\partial x = 2xy^3z^4 \qquad \partial f/\partial y = 3x^2y^2z^4 \qquad \partial f/\partial z = 4x^2y^3z^3$$

STEP 2: Use equation (1) to compute the gradient

$$\textbf{grad } f = \nabla f = 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$$

***Further* EXAMPLE:**

If you want to know the value of the gradient at any point (x, y, z), you just substitute the values of x, y, and z into the gradient formula.

Consider the function in the previous example, and let x = 1, y = 2, and z = 3. Compute the value of the gradient of that function at that point.

$$\textbf{grad } f = \nabla f = 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$$

$$= 2(1)(2^3)(3^4)\,\mathbf{i} + 3(1^2)(2^2)(3^4)\,\mathbf{j} + 4(1^2)(2^3)(3^3)\,\mathbf{k}$$

$$= 2(1)(8)(81)\,\mathbf{i} + 3(1)(4)(81)\,\mathbf{j} + 4(1)(8)(27)\,\mathbf{k}$$

$$= 1296\,\mathbf{i} + 972\,\mathbf{j} + 864\,\mathbf{k}$$

Interpretation

The gradient of a function at any point represents how quickly the function is changing, and points in the direction of the steepest (most rapid) change.